Deptt- MATHEMATICS

College- SOGHRA COLLEGE,BIHAR SHARIF

Part- BSc PART 2

Solved Examples

Example 1: Find the differential equation of the family of curves $y = Ae^x + Be^{-x}$

Sol: By differentiating the given equation twice, we will get the result.

$$\frac{dy}{dx} = Ae^x - Be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = Ae^x + Be^{-x} = y$$

Example 2: Find differential equation of the family of curves $y = c(x - c)^2$, where c is an arbitrary constant.

Sol: By differentiating the given family of curves and then eliminating c we will get the required differential equation.

$$y = c(x - c)^2$$

$$\Rightarrow \frac{dy}{dx} = 2(x - c)c$$

By division, $\frac{x-c}{2} = \frac{y}{dy/dx}$

or $c = x - \frac{2y}{dy/dx}$

Eliminating c, we get

$$\left(\frac{dy}{dx}\right)^2 = 4c^2(x - c)^2 = 4cy$$

$$= 4y\,4y\left\{x - \frac{2y}{dy/dx}\right\}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^3 = 4y\left[x\frac{dy}{dx} - 2y\right]$$

Example 3: Find the differential equation of all parabolas which have their vertex at (a, b) and where the axis is parallel to x-axis.

Sol: Equation of parabola having vertex at (a, b) and axis is parallel to x-axis is $(y - b)^2 = 4L(x - a)$ where L is a parameter. Hence by differentiating and eliminating L we will get required differential equation.

$$\therefore \quad 2(y - b)\frac{dy}{dx} = 4L$$

On eliminating L, we get

$$(y - b)^2 = 2(y - b)\ \frac{xdy - ydx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)\ (x - a)$$

Differential equation is,

$$2(x - a)\ \frac{dy}{dx} = y - b.$$

Example 4: Show that the function $y = be^x + ce^{2x}$ is a solution of the differential equation.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Sol: Differentiating given equation twice we can obtain the required differential equation.

$$y = be^x + ce^{2x}$$

$$\Rightarrow \frac{dy}{dx} = be^x + 2ce^{2x} = y + ce^{2x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = be^x + 4ce^{2x} = y + 3ce^{2x}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Example 5: Solve: $\frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = 0$

Sol: By separating x and y term and integrating both sides we can solve it.

$$\int\frac{dy}{\sqrt{1-y^2}} = -\int\frac{dx}{\sqrt{1-x^2}}$$

$\Rightarrow \sin^{-1}y = -\sin^{-1}x + c$ or $\sin^{-1}y + \sin^{-1}x = c$

Example 6: Find the equation of the curve that passes through the point P(1, 2) and satisfies the differential equation

$$\int f(x)dx + C = \frac{-2xy}{x^2+1} : y > 0$$

Sol: By integrating both sides we will get general equation of curve and then by substituting point (1, 2) in that we will get value of constant part.

$$\frac{dy}{dx} = -\frac{2xy}{x^2+1} \Rightarrow \int\frac{dy}{y} = -\int\frac{2x}{x^2+1}dx$$

$$\Rightarrow \log|y| = -\log(x^2 + 1) + \log c_0$$

$\Rightarrow \quad \log(|y|\,(x^2 + 1)) = \log c_0$

$\Rightarrow \quad |y|(x^2 + 1) = c_0$

As point P(1, 2) lies on it,

$2(1 + 1) = c_0$ or $c_0 = 4$

$\therefore \quad$ Curve is $y(x^2 + 1) = 4$

Example 7: Solve: $\frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5}$

Sol: By putting $x - y = t$ and integrating both sides we will obtain result.

Put $x - y = t$; then, $1 - \frac{dy}{dx} = \frac{dt}{dx}$

Differential equation becomes

$1 - \frac{dy}{dx} = \frac{t+3}{2t+5}$ or $\frac{dt}{dx} = 1 - \frac{t+3}{2t+5} = \frac{t+2}{2t+5}$

$\Rightarrow \int dx = \int \frac{2t \;\; +5}{t+2} dt = 2t + \int \frac{dt}{1+2}$

$\Rightarrow x + c = 2t + \log|t + 2| = 2(x - y) + \log|(x - y + 2)|$

Example 8: $x^2dy + y(x + y)dx = 0$: $xy > 0$

Sol: We can write the given equation as

$\left(\frac{dy}{dx}\right)^3 = 4y\left[x\frac{dy}{dx} - 2y\right] = -\frac{dy}{dx}$

and then by substituting $y = vx$ and integrating we will

get required general equation.

$\frac{dy}{dx} = -\frac{y(x+y)}{x^2}$ (Put $y = vx$)

$\Rightarrow v + x\frac{dy}{dx} = -v(1 + v)$

$\Rightarrow \frac{dv}{dx} = -2v - v^2$ or $\int \frac{dv}{v(v+2)} = \int \frac{dx}{x}$

$\Rightarrow -\int \frac{dx}{x} = \frac{1}{2}\int\left(\frac{1}{v} - \frac{1}{v+2}\right)dv$

$\Rightarrow -\log|x| = \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ $(\log|v| - \log|v+2| + c_0$

or $\left|\frac{V}{V+2}x^2 = c\right|$ or $\frac{\frac{y}{x}x^2}{\frac{y}{x}+2} = c$ or $\frac{yx^2}{y+2x} = c$

Example 9: Solve: $\frac{dy}{dx} + \sec x = \tan x : 0 < x < \frac{dy}{dx}$

Sol: The given equation is in the form of $\frac{dy}{dx} + px = q$ hence by using integration factor method we can solve it.

I.F. $= \frac{-2xy}{x^2 = 1} = \frac{dy}{dx} = -\frac{2xy}{x^2+1}$

$= \sec x + \tan x$

Solution is $y(\sec x + \tan x)$

$= \int \tan x(\sec x + \tan)dx$

$= \int \sec x \tan x dx + \int(\sec^2 x - 1)dx$

$= \sec x + \tan x - x + c$

or $(y - 1)(\sec x + \tan x) = c - x$

Example 10: Solve:

$\sin x.\cos y.dx + \cos x.\sin y.dy = 0$

given, $y = \frac{\pi}{4}$ when $x = 0$.

Sol: Here by separating variables and taking integration we will get the general equation and then using the given values of x and y we will get value of constant c.

We have,

$\sin x.\cos y.dx + \cos x.\sin y.dy = 0$

On separating the variables, we get

$\Rightarrow \frac{dt}{dx}$

Integrating both sides, we get

$\int \frac{\sin x}{\cos x}dx + \int \frac{\sin y}{\cos y}dy = 0$

[Dividing by cosx cosy], we get

$\Rightarrow \log|\sec x| + \log|\sec y| = \log C$

$\Rightarrow \log|\sec x|\,|\sec y| = \log C$

$\Rightarrow \sec x.\sec y = C$ ... (i)

On putting $y = \frac{\pi}{4}$, $x = 0$ in (i),

we have $C = \sec 0.\sec\frac{\pi}{4}$

$\Rightarrow C$ (1). $(\sqrt{2}) = \sqrt{2}$

Substituting the value of C in (i) we get

$\sec x.\frac{1}{\cos y} = \sqrt{2} \Rightarrow \cos y = \frac{1}{\sqrt{2}}\sec x$

$\Rightarrow y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\sec x\right)$

Example 11: Solve the differential equation

$\frac{dy}{dx} = x^2e^{-3y}$, given that y =0 for x =0.

Sol: Similar to the problem above we can solve it.

Here, $\frac{dy}{dx} = x^2e^{-3y}$ ..... (i)

On separating the variables, we have

$\Rightarrow \quad e^{3y}dy=x^2dx$

Integrating both sides, we get

$\int e^{3y} = \int x^2dx$

$\Rightarrow \quad \frac{e^{3y}}{3} = \frac{x^3}{3} + c$ ..... (ii)

putting: y = 0 for x = 0, in (ii), we obtain

$\frac{e^0}{3} = 0 + C \Rightarrow \frac{1}{3} = C \qquad [e^0 = 1]$

On substituting the value of C in (ii), we get

$\therefore \quad e^{3y} = x^3 + 1$

which is the required particular solution of (i)

Example 12: Solve the following differential equation:

$2x^2\frac{dy}{dx} - 2xy + y^2 = 0$

Sol: Here by rearranging the given equation we will get $e^{\log(\sec x+\tan x)} = \int \tan x(\sec x+\tan x)dx$. Now by substituting y = vx and then integrating we can solve the illustration above.

$2x^2\frac{dy}{dx} = 2xy - y^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy-y^2}{2x^2}$ ..... (i)

Put y = vx so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{2x(vx)-(vx)2}{2x^2}$

$\Rightarrow \quad v + x\frac{dv}{dx} = v - \frac{v^2}{2}$

$\frac{xdv}{dx} = \frac{-v^2}{2} \Rightarrow \frac{dv}{v^2}\frac{dx}{2x}$

Integrating, we have

$\Rightarrow \quad \frac{1}{4} = \frac{1}{4}|\log x| + c$

$\Rightarrow \frac{x}{y} = \frac{1}{2}|\log x| + c$

Example 13: Solve the following differential equation

$\cos^2x\frac{dy}{dx} + y = \tan x$

Sol: Here by reducing the given equation in the form of $\frac{dy}{dx} + py = q$ and then using integration factor we will get the result.

We have, $\cos^2x\frac{dy}{dx} + y = \tan x$

$\Rightarrow \frac{dy}{dx} + y.\sec^2x = \tan x.\sec^2x$

I.F. $= e^{\int \sec^{2x}} = e^{\tan x}$

$y.(\text{I.F.}) = \int Q.(\text{I.F.})dx + c$

$y.e^{\tan x} = \int \tan x \sec^2 x e^{\tan x}dx = \int te^t dt$

$te^t - \int e^t dt + c \quad \begin{bmatrix} \tan x = t \\ \sec^2 x dx = dt \end{bmatrix}$

$= te^t - e^t + c$

$= \tan x\, e^{\tan x} - e^{\tan x} + c$

$y = \tan x - 1 + ce^{-\tan x}$

Example 14: Solve $x\frac{dy}{dx} - y = x^2$

Sol: As similar to the problem above, we can reduce the given equation as $\frac{dy}{dx}$ therefore by using integration factor we can solve this.

We have, $x\frac{dy}{dx} - y = x^2$

$\Rightarrow \quad \frac{dy}{dx} - \frac{1}{x}y = x$ ... (i)

This is a linear differential equation in y

Here, $P = -\frac{2xy-y^2}{2x^2}$ and Q = x

Now, $\quad$ I.F. $= e^{\int pdx} = e^{\int -\frac{1}{x}dx}$

$= e^{-\log x} = e^{\log^{-1}} = x^{-1} = \frac{1}{x}$

$\therefore$ The solution of (i) is

$y(\text{I.F.}) = \int (Q \times \text{IF.})dx + C = x + C$

$\Rightarrow \quad y = x^2 + Cx$

Example 15: Solve the following differential equation:

$$\frac{dv}{dx} + y = \cos x - \sin x.$$

Sol: Here given equation is in the form of $\frac{dy}{dx} + Py = Q$, where P = 1 and Q = cosx − sinx hence by using integration factor we will get result.

Given differential equation is

$$\frac{dy}{dx} + y = \cos x - \sin x \quad \text{... (i)}$$

The given differential equation is a linear differential equation

On comparing with $\frac{dy}{dx} + Py = Q$

$\therefore \quad P = 1, Q = \cos x - \sin x$

$$\text{I.F.} = e^{\int p dx} = e^x$$

$\therefore$ required solution of (i) is

$$y\,(\text{I.F.}) = \int Q.(\text{I.F})dx + c$$

$$\Rightarrow y.e^x = \int (\cos x - \sin x)e^x dx + c$$

$$\Rightarrow y.e^x = \int \cos x e^x dx - \int \sin x.e^x dx + c$$

Integrating by parts, we get

$$\Rightarrow y.e^x = \cos x \int e^x dx - \int (-\sin x)e^x dx - \int \sec^2 x dx + c$$

$$\Rightarrow y.e^x = e^x \cos x + \int e^x \sin x dx - \int e^x \sin x dx + c$$

$$y.e^x = e^x \cos x + c$$

$$\therefore \quad y = \cos x + ce^{-x}$$

Example 1: Solve

$$\left(xe^{y/x} - y\sin\frac{y}{x}\right)dx + x\sin\frac{y}{x}dy = 0; x>0$$

Sol: Simply by putting y = vx and integrating we can solve the problem above.

$$\left(e^{\frac{y}{x}} - \frac{y}{x}\sin\frac{y}{x}\right) + \sin\frac{ydy}{xdx} = 0$$

Put y = vx

$$\therefore (e^v - v\sin v) + \sin v\left(v + x\frac{dv}{dx}\right) = 0$$

$$\Rightarrow \int \frac{dx}{x} + \int e^{-v}\sin v dv = 0$$

Integrating, we get

$$\log x - \frac{1}{2}e^{-v}(\sin v + \cos v) = c$$

$$\text{or } \log x = c + \frac{1}{2}e^{-y/x}.\left(\sin\frac{y}{x} - 4\cos\frac{y}{x}\right)$$

Example 2: Solve:

$$xdy - ydx = xy^3(1 + \log x)dx$$

Sol: We can reduce the given equation in the form of $-\frac{x}{y}d\left(\frac{x}{y}\right) = x^2(1 + \log x)dx$. Hence by integrating L.H.S. with respect to $\frac{x}{y}$ and R.H.S. with respect to x we will get the solution.

$$-\frac{ydx - xdy}{y^2} = xy(1 + \log x)dx$$

$$\text{or } -d\left(\frac{x}{y}\right) = xy\,(1 + \log x)dx$$

$$\text{or } -\frac{x}{y}d\left(\frac{x}{y}\right) = x^2(1 + \log x)dx$$

$$\text{Integrating, } -\int \frac{x}{y}d\left(\frac{x}{y}\right)$$

$$= \int x^2(1 + \log x)dx$$

$$\text{or } -\frac{1}{2}\left(\frac{x}{y}\right)^2 = (1 + \log x)\frac{x^3}{3} - \int \frac{x^3}{3}.\frac{1}{x}dx$$

$$\frac{-1}{2}\left(\frac{x}{y}\right)^2 = (1 + \log x)\frac{x^3}{3} - \frac{x^3}{9} + c$$

Example 3: Find the equation of the curve passing through (1, 2) whose differential equation is

$$y(x + y^3)dx = x(y^3 - x)dy$$

Sol: Similar to example 2 we can solve the problem above by reducing the given equation as –

$$\frac{y}{x}d\left(\frac{y}{x}\right) + \frac{1}{x^2y^2}d(xy) = 0.$$

$$(xy + y^4)dx = (xy^3 - x^2)dy$$

$$\text{or } y^3(ydx - xdy) + x(ydx + xdy) = 0$$

$$\text{or } -x^2y^3\frac{xdy - ydx}{x^2} + xd(xy) = 0$$

or $-\frac{y}{x}d\left(\frac{y}{x}\right)+\frac{1}{x^2y^2}d(xy)=0$

Integrating, we get

$$-\frac{1}{2}\left(\frac{y}{x}\right)^2-\frac{1}{xy}=c$$

or $y^3 + 2x - 2cx^2y = 0$

As it passes through (1, 2), condition is

$8 + 2 + 4c = 0 \Rightarrow c = -\frac{5}{2}$

Thus curve is $y^3 + 2x - 5x^2y = 0$

Example 4: Form the differential equation representing the family of curves $y = A\cos 2x + B\sin 2x$, where A and B are arbitrary constants.

Sol: Here we have two arbitrary constants hence we have to differentiate the given equation twice.

The given equation is:

$y = A\cos 2x + B\sin 2x$ ... (i)

Diff. w.r.t. x,

$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$

Again diff. w.r.t. x, $\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x$

$= -4(A\cos 2x + B\sin 2x) = -4y$ [Using (i)]

Hence $\frac{d^2y}{dx^2} + 4y = 0$, which is the required differential equation.

Example 5: The solution of the differential equation $x\frac{d^2y}{dx^2} = 1$, given that $y = 1, \frac{dy}{dx} = 0$, when $x = 1$, is

Sol: By integrating $x\frac{d^2y}{dx^2} = 1$ twice we will get its general equation and then by substituting given values of x, y and $\frac{dy}{dx}$ we will get the values of the constants.

$$x\frac{d^2y}{dx^2} = 1 \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \log x + C_1$$

Again integrating

$y = x\log x - x + C_1x + C_2$

Given $y = 1$ and $\frac{dy}{dx} = 0$ at $x = 1$

$\Rightarrow C_1 = 0$ and $C_2 = 2$

Therefore, the required solution is $y = x \log x - x + 2$

Example 6: By the elimination of the constant h and k, find the differential equation of which $(x-h)^2+(y-k)^2=a^2$, is a solution.

Sol: Three relations are necessary to eliminate two constants. Thus, besides the given relation we require two more and they will be obtained by differentiating the given relation twice successively.

Thus we have

$(x-h) + (y-k)\frac{dy}{dx} = 0$ ... (i)

$1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ ... (ii)

From (i) and (ii), we obtained

$$y-k = -\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

$$x-h = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

Substitute these values in the given relation, we obtained

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

Example 7: Form the differential equations by eliminating the constant(s) in the following problems.

(a) $x^2 - y^2 = c(x^2 + y^2)^2$, (b) $a(y + a)^2 = x^3$

Sol: Given equations have one arbitrary constant, hence by differentiating once and eliminating c and a we will get the required differential equation.

(a) The given equation contains one constant

Differentiating the equation once, we get

$2x - 2yy' = 2c(x^2 + y^2)(2x + 2yy')$

But $c = \dfrac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$

Substituting for c, we get

$$(x - yy') = \frac{\left(x^2 + y^2\right)\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} .2(x + yy')$$

or $(x^2 + y^2)(x - yy') = 2(x^2 - y^2)(x + yy')$

$\Rightarrow yy'[(x^2 + y^2) + 2(x^2 - y^2)]$

$\Rightarrow x(x^2 + y^2) - 2x(x^2 - y^2)$

$\Rightarrow yy'(3x^2 - y^2) = x(3y^2 - x^2)$

Hence, $y' = \dfrac{x\left(3y^2 - x^2\right)}{y\left(3x^2 - y^2\right)}$

(b) The given equation contains only one constant. Differentiating once, we get

$2a(y + a)y' = 3x^2$ ... (i)

Multiplying by y + a, we get

$2a(y + a)^2 y' = 3x^2(y + a)$

Using the given equation, we obtain

$2x^3 y' = 3x^2(y + a)$ or $2xy' = 3y + 3a$

or $a = \dfrac{1}{3}(2xy' - 3y)$

Substituting the value of a in (i) we obtain

$$\frac{2}{3}(2xy' - 3y)\left[y + \frac{1}{3}(2xy - 3y)\right]y' = 3x^2$$

$$\frac{2}{9}(2xy' - 3y)(2xy')y' = 3x^2$$

Cancelling x, we obtain

$8x(y')^3 - 12y(y')^2 - 27x = 0$

Example 8: If $y(x - y)^2 = x$, then show that

$$\int \frac{dx}{(x - 3y)} = \frac{1}{2}\log[x - y)^2 - 1]$$

Sol: As given $y(x - y)^2 = x$, therefore by differentiating it with respect to x we will get the value of $\frac{dy}{dx}$. After that differentiate both sides of equation $\int \frac{dx}{(x - 3y)} = \frac{1}{2}$ $\log[x - y)^2 - 1]$ w.r.t. x and then by substituting the value of $\frac{dy}{dx}$ we can prove it.

Let $P = \int \dfrac{dx}{(x - 3)} = \dfrac{1}{2}\log\{(x - y)^2 - 1\}$

$\therefore \quad P = \int \dfrac{dx}{(x - 3y)}$

$\dfrac{dP}{dx} = \dfrac{1}{(x - 3y)}$ ... (i)

Also $P = \dfrac{1}{2}\log\{(x - y)^2 - 1\}$

$\therefore \dfrac{dP}{dx} = \dfrac{(x - y)\left\{1 - \dfrac{dy}{dx}\right\}}{\left\{(x - y)^2 - 1\right\}}$ ... (ii)

Given $y(x - y)^2 = x$

Differentiating both sides w.r.t. x

$\therefore \dfrac{dy}{dx} = \dfrac{1 - 2y(x - y)}{(x - y)(x - 3y)}$ ... (iii)

From (ii) and (iii)

$$\frac{dP}{dx} = \frac{(x - y)\{1 - (1 - 2y(x - y)/(x - y)(x - 3y))\}}{\{(x - y)^2 - 1\}}$$

$$= \frac{(x - y)(x - 3y) - 1 + 2y(x - y)}{(x - 3y)\{(x - y^2) - 1\}}$$

$$= \frac{\{(x - y)^2 - 1\}}{(x - 3y)\{(x - y)^2 - 1\}}$$

$$\Rightarrow \frac{dP}{dx} = \frac{1}{(x - 3y)}$$

It is true from (i)

Hence $\displaystyle\int \frac{dx}{(x - 3y)} = \frac{1}{2}\ell n\left\{(x - y)^2 - 1\right\}$

Example 9: Solve: $\cos(x + y)dy = dx$

Sol: Simply by putting $x + y = t$ we can reduce the given equation as $\frac{dt}{dx} = \sec t + 1$ and then by separating the variable and integrating we can solve the problem given above.

We have $\cos(x + y)dy = dx$

$\Rightarrow \dfrac{dy}{dx} = \sec(x + y)$

On putting x + y = t so that $1+\frac{dy}{dx}=\frac{dt}{dx}$

or $\frac{dy}{dx}=\frac{dt}{dx}-1$ we get

$\frac{dt}{dx}-1=\sec$

$\Rightarrow \frac{dt}{dx}=1+\sec t$

$\frac{dt}{1+\sec t}=dx \Rightarrow \frac{\cos t}{\cos t+1}dt=dx$

$\int\frac{\cos t}{\cos t+1}dt=\int dx$

$\Rightarrow \int\left[1-\frac{1}{\cos t+1}\right]' dt = x + C$

$\int\left[1-\frac{1}{2\cos^2(t/2)-1+1}\right] dt = x + C$

$\int\left(1-\frac{1}{2}\sec^2\frac{t}{2}\right)dt = x + C$

$\Rightarrow t-\tan\frac{t}{2} = x + C$

$x + y-\tan\frac{x+y}{2} = x + C$

$y-\tan\frac{x+y}{2} = C$

Example 10: Solve: $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Sol: Similar to example 9.

We have, $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y \Rightarrow \frac{dy}{dx} = \sin(x + y)$

Putting x + y = t, so that

$1+\frac{dy}{dx}=\frac{dt}{dx} \Rightarrow \frac{dy}{dx}=\frac{dt}{dx}$

Now, substituting x + y = t and $\frac{dy}{dx}=\frac{dt}{dx}-1$ in (i), we get

$\frac{dt}{dx} = \sin t \Rightarrow \frac{dt}{dx} = \sin t + 1 \Rightarrow dx = \frac{dt}{1+\sin t}$

Integrating both sides, we get

$\int dx=\int\frac{dt}{1+\sin^2 t}dt+c$

$\Rightarrow \int dx=\int\frac{1-\sin t}{1-\sin^2 t}dt + C = \int\frac{1-\sin t}{\cos^2 t}dt$

$\Rightarrow \int dx=\int\left(\sec^2 t-\tan t\sec t\right)dt$

$\Rightarrow x = \tan t - \sec t$

$\Rightarrow x = \tan(x + y) - \sec(x + y) + C$

Example 11: Solve the equation:

$\frac{dy}{dx}=\frac{y}{x}+x\sin\frac{y}{x}$

Sol: Simply by putting y = vx and integrating we can obtain the general equation of given differential equation.

We have,

$\frac{dy}{dx}=\frac{y}{x}+x\sin\frac{y}{x}$ ... (i)

Put y = vx, so that

$\frac{dy}{dx}=v+x\frac{dv}{dx}$

On putting the value of y and $\frac{dy}{dx}$ in (i), we get

$v + x\frac{dv}{dx} = v + x\sin v$

$\Rightarrow \quad x\frac{dv}{dx} \Rightarrow \frac{dv}{dx}=\sin v$

Separating the variables, we get

$\frac{dv}{\sin v}=dx \Rightarrow \int\operatorname{cosec}v\, dv=\int dx$

$\Rightarrow \log\tan\frac{v}{2} = x + C$ ... (ii)

On putting the value of v in (ii), we have

$\log\tan\frac{y}{2x} = x + C$

This is the required solution

Example 12: Solve:

$2ye^{\frac{x}{y}}dx+\left(y-2xe^{\frac{x}{y}}\right)dy=0$

Sol: We can reduce the given equation as $\frac{dy}{dx}=\frac{2xe^{x/y}}{2ye^{x/y}}$ and then by putting x = vy and integrating we can obtain general equation.

We have,

$2ye^{\frac{x}{y}}dx+\left(y-2xe^{\frac{x}{y}}\right)dx=0$

$$\Rightarrow \quad 2ye^{\frac{x}{y}}.\frac{dx}{dy}+\left(y-2xe^{\frac{x}{y}}\right)=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2xe^{x/y}}{2ye^{x/y}} \qquad \text{... (i)}$$

Clearly, the given differential equation is a homogeneous differential equation. As the right hand side of (i) is expressible as a function of $\left(\frac{x}{y}\right)$. So, we put

$\frac{dt}{dx}=v \Rightarrow x = vy$ and $\frac{dx}{dy}$

$= v + y\frac{dv}{dy}$ in (i), we get

$$v+y\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}}$$

$$\Rightarrow \quad y\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}}-v$$

$$\Rightarrow y\frac{dv}{dy}=-\frac{1}{2e^{v}}$$

$$\Rightarrow \quad 2ye^{v}dv = -dy$$

$$\Rightarrow 2e^{v}dv = -\frac{1}{y}dy \ , y \neq 0$$

Integrating both sides, we get

$$2\int e^{v}dv=-\int\frac{1}{y}dy+\log c$$

$$\Rightarrow \quad 2e^{v} = -\log|y| + \log c$$

$$\Rightarrow \quad 2e^{v} = \log\left|\frac{c}{y}\right|$$

$$\Rightarrow \quad 2e^{\frac{x}{y}}=\log\left|\frac{c}{y}\right| \quad \left(\because v=\frac{x}{y}\right)$$

Example 13: Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2+y^2}{2xy}$, is given by $x^2 - y^2 = cx$

Sol: Here by reading the above problem, we get that $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$. Hence by putting $y = vx$ and then integrating both sides we can prove the given equation.

We have slope of the tangent

$$=\frac{x^2+y^2}{2xy} \Rightarrow \frac{dy}{dx}=\frac{x^2+y^2}{2xy}$$

$$\text{or} \quad \frac{dy}{dx}=\frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}} \qquad \text{... (i)}$$

Equation (i) is a homogeneous differential equation.

So we put $y = vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

Substituting the value of $\frac{y}{x}$ and $\frac{dy}{dx}$ in equation (i), we get

$$v + x\frac{dv}{dx}=\frac{1+v^2}{2v}$$

$$\text{or} \quad x\frac{dv}{dx}=\frac{1-v^2}{2v} \qquad \text{... (ii)}$$

Separating the variables in equation (ii), we get

$$\frac{2v}{1-v^2}dv=\frac{dx}{x} \text{ or } \frac{2v}{v^2-1}dv=-\frac{dx}{x} \qquad \text{... (iii)}$$

Integrating both sides of equation (iii), we get

$$\int\frac{2v}{v^2-1}dv=-\int\frac{1}{x}dx$$

$$\text{or} \quad \log|v^2-1| = -\log|x| + \log|C_1|$$

$$\text{or} \quad \log|(v^2-1)(x)| = \log|C_1| \qquad \text{... (iv)}$$

Replacing v by $\frac{y}{x}$ in equation (iv), we get $\left(\frac{y^2}{x^2}-1\right)$ $x = \pm C_1$

or $(y^2 - x^2) = \pm C_1x$ or $x^2 - y^2 = Cx$

Example 14: Solve: $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Sol: Simply by putting $x = X + h$; $y = Y + k$ were (h, k) will satisfy the equations $x + 2y - 3 = 0$ and $2x + y - 3 = 0$ we can solve the problem.

$$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$$

Put: $x = X + h$; $y = Y + k$

$$\Rightarrow \quad dx = dX;\ dy = dY$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{dY}{dX}$$

Given equation reduces to

$$\frac{dy}{dx}=\frac{(x+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3}=\frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \qquad \text{... (i)}$$

Choose h and k such that

$h + 2k - 3 = 0$; and $2h + k - 3 = 0$

$\Rightarrow \quad h = 1 ; k = 1$

Equation (i) becomes

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y} \qquad \text{... (ii)}$$

Put: $Y = VX$

$$\Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$$

Now equation (ii) becomes:

$$V + X\frac{dV}{dX} = \frac{X+2VX}{2X+VX} = \frac{1+2V}{2+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+2V}{2+V} - V = \frac{1-V^2}{2+V}$$

Separating the variables, we have

$$\Rightarrow \quad \frac{2+V}{1-V^2}dV = \frac{dX}{X}$$

Integrating, we get

$$\int\frac{2}{1-V^2}dV + \int\frac{V}{1-V^2}dV = \int\frac{dX}{X}$$

$$\Rightarrow 2.\frac{1}{2}\log\frac{1+V}{1-V} - \frac{1}{2}\log(1-V^2) = \log X + \log c$$

$$\Rightarrow 2\log\left(\frac{1+V}{1-V}\right) - \log(1-V^2) = 2\log cX$$

$$\Rightarrow \log\left[\left(\frac{1+V}{1-V}\right)^2 \times \frac{1}{1-V^2}\right] = \log(cX)^2$$

$$\Rightarrow \left(\frac{1+V}{1-V}\right)^2 \times \frac{1}{(1-V^2)} = (cX)^2$$

$$\Rightarrow \frac{1+V}{(1-V)^3} = c^2X^2 \qquad \text{... (iii)}$$

Putting the value of V in (iii), we have

$$\Rightarrow \frac{X+Y}{(X-Y)^3}X^2 = c^2X^2 \qquad \text{... (iv)}$$

$$\left[\because V = \frac{Y}{X}\right]$$

Substituting the value of X and Y in (iv), we get

$$\Rightarrow \frac{x+y-2}{(x-y)^3} = c^2$$

$[\because X = x - h = x - 1; Y = y - 1]$

Example 15: Solve the following differential equation:

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$$

Sol: First reduce this into the form of $\frac{dy}{dx} + Py = Q$ and then using the integration factor i.e. $e^{\int Pdx}$ we can solve this.

We have, $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{2}{(x^2-1)^2} \qquad \text{.....(i)}$$

This is a linear differential equation in y.

Here, $P = \frac{2x}{x^2-1}$, $Q = \frac{2}{(x^2-1)^2}$

$$\text{I.F.} = e^{\int Pdx} = e^{\int\frac{2x}{x^2-1}dx} = e^{\log(x^2-1)} = x^2 - 1$$

$\therefore$ The solution of (i) is

$$y(\text{I.F.}) = \int(Q \times \text{I.F})dx + C$$

$$y.(x^2 - 1) = \int(x^2-1).\frac{2}{(x^2-1)^2}dx + C$$

$$= 2\int\frac{1}{x^2-1}dx + C$$

$$= 2.\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

$$\Rightarrow \quad y(x^2 - 1) = \log\left|\frac{x-1}{x+1}\right| + C$$

$$\Rightarrow \quad y = \left(\frac{1}{x^2-1}\right)\left[\log\left|\frac{x-1}{x+1}\right| + C\right]$$